

# BFFT quantization and dynamical solutions of a fluid field theory

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We study a field theory formulation of a fluid mechanical model. We implement the Hamiltonian formalism by using the BFFT conjecture in order to build a gauge invariant fluid field theory. We also generalize previous known classical dynamical field solutions for the fluid model.

## I. INTRODUCTION

The basis of the canonical quantization for systems with infinite degrees of freedom has been the powerful character and applicability of the Dirac method [1]. Despite to current use in different systems, alternative formalisms have been developed in order to solve particular difficulties which come from Dirac's formulation [2]. One of these problems is the role of the first- and second-class constraints when we identify the classical brackets as commutators. While first-class constraints are related to symmetries the second-class ones may imply some ambiguities when treated as quantum operators. The physical status of a theory is chosen by imposing complementary conditions which are given by the first-class constraints. In order to avoid the presence of second-class constraints we can separate it into first-class ones and gauge fixing terms, however there is a special situation where the constraints are nonlinear so that this procedure fails [3].

An alternative way to circumvent this difficulty is to employ an interesting machinery proposed by Batalin, Fradkin, Fradkina and Tyutin (BFFT) [4], which converts the second-class constraints into first order ones by using auxiliary fields. Its applicability has been demonstrated in many different systems involving linear constraints [5,6] and also in nonlinear cases [3,7,8]. As we expect, the implementation of the above mentioned method through the introduction of new fields gives rise to a kind of Wess-Zumino terms which turns the resulting effective theory gauge invariant.

In this paper we discuss the Hamiltonian formalism for a scalar field fluid theory from BFFT method point of view. The fluid field theory has been introduced as

a laboratory to study some classical aspects of membrane problem [9] but there are also other classical and quantum systems which can be described by this model [10–12]. We can mention, for instance, the hydrodynamical formulation of quantum mechanics [13] or a dimensional reduction of a relativistic scalar field theory [14].

Recently, Bazeia and Jackiw [10] have discussed this model by making a careful analysis of the Galileo and Poincaré symmetries. In particular, they obtained dynamical solutions for the original fields by choosing a singular potential (see also [11,12]). In our study we use the BFFT method to build a gauge invariant theory to obtain the respective generators of the extended gauge transformations. As a consequence of this symmetry we show that for linear constraints, the Lagrangian is invariant in a similar way in respect to that discussed by Amorim and Barcelos [6] for chiral bosons theories.

We have organized this paper as follows: In section II we present the fluid field theory as described by Bazeia and Jackiw. We show that their dynamical solutions for the singular potential can be generalized to other potentials leading to diverse physical systems. Section III is dedicated to the explanation of the BFFT method applied to the fluid field theory. Finally in section IV we present an analysis of the results obtained and give our conclusions. We have also included two appendices where some technical calculations are given.

## II. THE MODEL ITS SYMMETRIES AND SOLUTIONS

Let us consider a fluid dynamical model [15] described by the following Lagrangian in  $d$  dimensional  $\mathbf{r}$  space, evolving in time  $t$ :

$$L = \int d^d r \left( \theta \dot{\rho} - \frac{1}{2} \rho \nabla \theta \cdot \nabla \theta - V(\rho) \right) \quad (2.1)$$

where  $\rho = \rho(t, \mathbf{r})$ ,  $\theta = \theta(t, \mathbf{r})$  and the over dot means time differentiation. In a usual fluid mechanical model  $\rho$  is the mass density and  $\theta$  is the velocity potential,  $\mathbf{v} = \nabla \theta$ . Then, we have the equations of motion:

$$\dot{\rho} = \nabla \cdot (\rho \nabla \theta) \quad (2.2)$$

$$\dot{\theta} = -\frac{1}{2} (\nabla \theta)^2 - \frac{\delta}{\delta \rho} \int d^d r V \quad (2.3)$$

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These equations of motion are recognized as the conventional ones for isentropic irrotational fluids [15]. As this system is a nonrelativistic one it has naturally the Galilean symmetry [16], *i. e.*, it is invariant under the Galilean Group which generators are

$$H = \int d^d r \mathcal{E}; \quad \mathcal{E} = \frac{1}{2} \rho (\nabla \theta)^2 + V(\rho) \quad (2.4)$$

$$\mathbf{P} = \int d^d r \mathcal{P}; \quad \mathcal{P} = \rho \nabla \theta \quad (2.5)$$

$$J^{ij} = \int d^d r \mathcal{J}^{ij}; \quad \mathcal{J}^{ij} = r^i \mathcal{P}^j - r^j \mathcal{P}^i \quad (2.6)$$

$$\mathbf{B} = \int d^d r \mathcal{B}; \quad \mathcal{B} = t \mathcal{P} - \mathbf{r} \rho \quad (2.7)$$

$$N = \int d^d r \rho \quad (2.8)$$

which respectively give time and space translation and space rotation, in addition to the Galileo boost and “charge” generator  $N$  which in this case is the total mass of the fluid, being naturally conserved. The Poisson brackets of these generators close under an algebra corresponding to the Galileo group. For instance,

$$\{B^i, P^j\} = \delta^{ij} N. \quad (2.9)$$

A connection with the membrane problem [9] is done when  $d = 2$  and

$$V(\rho) = \frac{g}{\rho} \quad (2.10)$$

where  $g$  is the coupling constant (this potential also connects the fluid field theory problem to  $d$ -branes in  $d + 1$  space dimensions [11,12]). In this case, the action  $I = \int dt L$  is invariant under time rescaling  $t \rightarrow e^w t$  generated by (dilation)

$$D = \int d^d r (t \mathcal{E} - \rho \theta) \quad (2.11)$$

Another symmetry of this action is given implicitly by [10]

$$\begin{aligned} t \rightarrow T(t, \mathbf{r}) &= t + \mathbf{w} \cdot \mathbf{r} + \frac{1}{2} \mathbf{w}^2 \theta(\mathbf{T}, \mathbf{R}) \\ \mathbf{r} \rightarrow \mathbf{R}(t, \mathbf{r}) &= \mathbf{r} + \mathbf{w} \theta(T, R) \end{aligned}$$

where

$$\theta(T, R) = \theta(t, \mathbf{r} - \mathbf{w}t) + \mathbf{w} \cdot \mathbf{r} - \frac{1}{2} \mathbf{w}^2 t$$

which are generated by

$$\mathbf{G} = \int d^d r (\mathbf{r} \mathcal{E} - \theta \mathcal{P}). \quad (2.12)$$

The geometrical meaning of  $\mathbf{G}$  is not clear, as pointed out in Ref. [10], however one should note that  $D$  and  $\mathbf{G}$

depend on the velocity potential  $\theta$  which is meaningful only in the irrotational case. They speculated that, since the fluid field model corresponds to a gauge-fixed version of the relativistic membrane in the light-cone, the symmetry generated by  $\mathbf{G}$  may be a residual gauge invariance of that model. As is well known, the Galileo group (2.4)-(2.8) together with the generators (2.11) and (2.12) defined in  $(d+1)$  dimensions is isomorphic to a Poincaré group defined in  $(d+1, 1)$  dimensions [17].

Bazeia and Jackiw presented some solutions for this system with the potential (2.10) which are

$$\theta(t, \mathbf{r}) = -\frac{r^2}{2(d-1)t} \quad (2.13)$$

$$\rho(t, \mathbf{r}) = \sqrt{\frac{2g}{d}} (d-1) \frac{|t|}{r} \quad (2.14)$$

valid for  $d > 1$ . They discuss other solutions for the free case (a particular case of the above solution) and also some solutions for  $d = 1$ , which are not our main concern here. Other solutions in different dimensions can also be found in [11].

Here we note that it is possible to extend the above results considering now the following potential:

$$V(\rho) = \frac{g}{\rho^n} \quad (2.15)$$

which describe ideal polytropic gases, *i. e.*, a gas in which the pressure is proportional to a power of the density [18], for which we find the solutions

$$\theta(t, \mathbf{r}) = -\frac{r^2}{[d(n+1)-2]t} \quad (2.16)$$

$$\rho(t, \mathbf{r}) = \left\{ \frac{ng[d(n+1)-2]^2 t^2}{d(n+1)r^2} \right\}^{1/(n+1)} \quad (2.17)$$

which are valid in  $d > 1$  space dimensions, generalizing the solutions obtained by Bazeia and Jackiw [10]. Their solution is a particular case of the above class of solutions which can be recovered when we take  $n = 1$  in the above equations. As we are going to show in the following section, the above fluid field theory also admits a gauge symmetry which is respected by a general potential  $V(\rho)$ .

### III. BFFT QUANTIZATION AND GAUGE SYMMETRY

Let us now construct a gauge invariant version of the model described above using the method developed by Batalin, Fradkin, Fradkina, and Tuyutin (BFFT) [4] which transforms second-class constraints into first-class ones [6].

Considering the Lagrangian (2.1) we obtain the primary constraints

$$\chi_\rho = \Pi_\rho - \theta \approx 0 \quad (3.1)$$

$$\chi_\theta = \Pi_\theta \approx 0 \quad (3.2)$$

which satisfy the algebra

$$\begin{aligned} \{\chi_\rho, \chi_\theta\} &= -\epsilon_{\rho\theta} \delta(x-y) \\ &\equiv \Delta_{\rho\theta}(x, y) \end{aligned} \quad (3.3)$$

Then, the primary Hamiltonian is given by

$$\begin{aligned} H_p &= \int d^d r \left( \Pi_\rho \dot{\rho} + \Pi_\theta \dot{\theta} - L + \lambda_\rho \chi_\rho + \lambda_\theta \chi_\theta \right) \\ &= \int d^d r \left( (\Pi_\rho - \theta) \dot{\rho} + \Pi_\theta \dot{\theta} + \frac{1}{2} \rho (\nabla \theta)^2 \right. \\ &\quad \left. + V(\rho) + \lambda_\rho \chi_\rho + \lambda_\theta \chi_\theta \right) \\ &= \int d^d r \left( \frac{1}{2} \rho (\nabla \theta)^2 + V(\rho) + \tilde{\lambda}_\rho \chi_\rho + \tilde{\lambda}_\theta \chi_\theta \right) \end{aligned} \quad (3.4)$$

where we defined  $\tilde{\lambda}_\rho = \lambda_\rho + \dot{\rho}$ ,  $\tilde{\lambda}_\theta = \lambda_\theta + \dot{\theta}$ . The consistency condition for the constraints determine the fields  $\tilde{\lambda}_\rho$ ,  $\tilde{\lambda}_\theta$  and there are no other constraints.

Before we implement the BFFT method it is necessary here to make a brief review of it. For a more comprehensive and elegant discussion see Refs. [4,6]. Let us now begin by extending the phase space including the new fields  $\varphi_\rho$  and  $\varphi_\theta$  which satisfy the algebra

$$\{\varphi_\rho, \varphi_\theta\} = \omega_{\rho\theta}(x, y) \quad (3.5)$$

such that the new constraints  $\Omega_\rho$ ,  $\Omega_\theta$  should be of first-class and could be written in general as

$$\Omega_\beta = \chi_\beta + \sigma_{\beta\alpha} \varphi^\alpha \quad (3.6)$$

where  $\sigma_{\beta\alpha} = \sigma_{\beta\alpha}(\rho, \theta)$ . The central idea of the BFFT method is to write the first-class constraints in terms of the second-class ones as

$$\Omega_\beta = \sum_{n=0}^{\infty} \chi_\beta^{(n)}, \quad (3.7)$$

with the condition  $\chi_\beta^{(0)} \equiv \chi_\beta$ . So,  $\chi_\beta^{(n)}$  is of  $n$ th order in the field  $\varphi_\alpha$ . The new constraints defined by Eqs. (3.6)-(3.7) satisfy the relation  $\{\Omega_\alpha, \Omega_\beta\} = 0$  and then

$$\{\chi_\alpha, \chi_\beta\}_{(\rho, \theta)} + \{\chi_\alpha^{(1)}, \chi_\beta^{(1)}\}_{(\varphi)} = 0 \quad (3.8)$$

$$\begin{aligned} \{\chi_\alpha, \chi_\beta^{(1)}\}_{(\rho, \theta)} + \{\chi_\alpha^{(1)}, \chi_\beta\}_{(\rho, \theta)} \\ + \{\chi_\alpha^{(1)}, \chi_\beta^{(2)}\}_{(\varphi)} + \{\chi_\alpha^{(2)}, \chi_\beta^{(1)}\}_{(\varphi)} = 0 \end{aligned} \quad (3.9)$$

$$\begin{aligned} \{\chi_\alpha, \chi_\beta^{(2)}\}_{(\rho, \theta)} + \{\chi_\alpha^{(1)}, \chi_\beta^{(1)}\}_{(\rho, \theta)} \\ + \{\chi_\alpha^{(2)}, \chi_\beta\}_{(\varphi)} + \{\chi_\alpha^{(1)}, \chi_\beta^{(3)}\}_{(\varphi)} \\ + \{\chi_\alpha^{(2)}, \chi_\beta^{(2)}\}_{(\varphi)} + \{\chi_\alpha^{(3)}, \chi_\beta^{(1)}\}_{(\varphi)} = 0. \end{aligned} \quad (3.10)$$

Here, we are using the notation  $\{ , \}_{(\rho, \theta)}$ ,  $\{ , \}_{(\varphi)}$  referring to the Poisson brackets of the pairs  $(\rho, \theta)$  and

$(\varphi_\rho, \varphi_\theta)$ . From Eqs. (3.3), (3.5), (3.6), (3.8) and using that  $\{\Omega_\alpha, \Omega_\beta\} = 0$  we have

$$\Delta_{\rho\theta} = -\sigma_{\rho\alpha} \omega^{\alpha\beta} \sigma_{\theta\beta}, \quad (3.11)$$

which for the fluid field problem can be written as

$$\begin{aligned} \Delta_{\rho\theta} &= -\epsilon_{\rho\theta} \delta(x-y) \\ &= -\int dz dz' \sigma_{\rho\alpha}(x, z) \omega^{\alpha\beta}(z, z') \sigma_{\theta\beta}(y, z'). \end{aligned} \quad (3.12)$$

As  $\omega^{\alpha\beta}$  is obtained from second-class constraints  $\varphi_\alpha$  we can choose  $\omega^{\alpha\beta} = \epsilon_{\alpha\beta} \delta(x-y)$  which implies  $\sigma_{\rho\alpha} = \epsilon_{\rho\alpha} \delta(z-x)$  so that

$$\begin{aligned} \Omega_\rho(x) &= \chi_\rho(x) + \int dz \delta(z-x) \varphi_\theta(z) \\ &= \chi_\rho(x) + \sigma_{\rho\theta} \varphi^\theta(x) \end{aligned} \quad (3.13)$$

and similarly

$$\Omega_\theta(x) = \chi_\theta(x) + \sigma_{\theta\rho} \varphi^\rho(x) \quad (3.14)$$

The next step is to include corrections to the canonical Hamiltonian. We remark that in this formalism any dynamical function  $A(\rho, \theta)$  can also be properly modified in order to be strong involutive with first-order constraints. So, if  $\tilde{A}(\rho, \theta, \varphi)$  is this quantity we have

$$\{\Omega_\rho, \tilde{A}\} = 0 \quad (3.15)$$

with the boundary condition

$$\tilde{A}(\rho, \theta, 0) = A(\rho, \theta). \quad (3.16)$$

In order to generate  $\tilde{A}$  we can repeat the same steps for the obtainment of  $\Omega_\rho$  above, i. e. we consider the expansion

$$\tilde{A} = \sum_{n=0}^{\infty} A^{(n)}, \quad (3.17)$$

where  $A^{(n)}$  is a term of order  $n$  in the field  $\varphi$ . Consequently, from Eqs. (3.8)-(3.10) rewritten for  $A^{(n)}$  and the condition  $A^{(0)} = A$ , we have

$$A^{(1)} = -\varphi^\alpha \omega_{\alpha\beta} \sigma^{\beta\gamma} \{\chi_\gamma, A\}, \quad (3.18)$$

where  $\omega_{\alpha\beta} = (\omega^{\alpha\beta})^{-1}$  and  $\sigma^{\beta\gamma} = (\sigma_{\beta\gamma})^{-1}$ . An equation analogous to (3.9) for  $A^{(2)}$  gives

$$\{\chi_\rho^{(1)}, A^{(2)}\} = -G_\rho^{(1)} \quad (3.19)$$

such that

$$\begin{aligned} G_\rho^{(1)} &= \{\chi_\rho, A^{(1)}\}_{(\rho, \theta)} + \{\chi_\rho^{(1)}, A\}_{(\rho, \theta)} \\ &\quad + \{\chi_\rho^{(2)}, A^{(1)}\}_{(\varphi)}. \end{aligned} \quad (3.20)$$

Then, we have for  $A^{(2)}$

$$A^{(2)} = -\frac{1}{2}\varphi^\alpha \omega_{\alpha\beta} \sigma^{\beta\gamma} G_\gamma^{(1)}. \quad (3.21)$$

and in general for  $n \geq 1$

$$A^{(n+1)} = -\frac{1}{n+1}\varphi^\alpha \omega_{\alpha\beta} \sigma^{\beta\gamma} G_\gamma^{(n)}, \quad (3.22)$$

with the auxiliary condition  $G_\rho^{(0)} = \{\chi_\rho, A\}$  so that

$$\begin{aligned} G_\rho^{(n)} &= \sum_{m=0}^n \{\chi_\rho^{(n+m)}, A^{(m)}\}_{(\rho, \theta)} \\ &+ \sum_{m=0}^{n-2} \{\chi_\rho^{(n-m)}, A^{(m+2)}\}_{(\varphi)} \\ &+ \{\chi_\rho^{(n+1)}, A^{(1)}\}_{(\varphi)}. \end{aligned} \quad (3.23)$$

**A. Particular Case:**  $\sigma_{\alpha\beta}$  independent of  $(\rho, \theta)$

In the particular case where  $\sigma_{\alpha\beta}$  does not depend on  $(\rho, \theta)$  the  $A^{(n)}$  are still given by Eq. (3.22) but

$$G_\rho^{(n)} = \{\chi_\rho, A^{(n)}\}. \quad (3.24)$$

Let us analyze this particular case further with the additional hypothesis that the second-class constraints are all linear. Then, we can write

$$\begin{aligned} A^{(n+1)} &= -\frac{1}{n+1}\varphi^\alpha \omega_{\alpha\beta} \sigma^{\beta\gamma} \{\chi_\gamma, A^{(n)}\} \\ &= -\frac{1}{n+1}\varphi^\alpha \omega_{\alpha\beta} \sigma^{\beta\gamma} \{\chi_\gamma, z^i\} \frac{\partial}{\partial z^i} A^{(n)} \\ &= -\frac{1}{n+1}\varphi^\alpha k_\alpha^i \partial_i A^{(n)} \end{aligned} \quad (3.25)$$

where we have used the Jacobi identity and the defined  $z^i$  is a generalized phase space coordinate, in the sense that it could be either a canonical coordinate or a canonical momentum. Note that  $k_\alpha^i \equiv \omega_{\alpha\beta} \sigma^{\beta\gamma} \{\chi_\gamma, z^i\}$  is a constant matrix since the constraints are linear,  $\{\chi_\gamma, z^i\} = \text{constant}$  and  $\sigma^{\beta\gamma}$  is a constant matrix too in this particular case. Using Eq. (3.25) iteratively one finds

$$A^{(n)} = \frac{(-1)^n}{n!} (\varphi^\alpha k_\alpha^i \partial_i)^n A \quad (3.26)$$

so that

$$\begin{aligned} \tilde{A}(\rho, \theta; \varphi) &\equiv \sum_{n=0}^{\infty} A^{(n)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\varphi^\alpha k_\alpha^i \partial_i)^n A \\ &= \exp(\varphi^\alpha k_\alpha^i \partial_i) A \end{aligned} \quad (3.27)$$

and then in this case the operator  $\tilde{A}$  will be of the form

$$\tilde{A}(z^i, \varphi^\alpha) = A(z^i - \varphi^\alpha k_\alpha^i). \quad (3.28)$$

**B. General Case:**  $\sigma_{\alpha\beta}$  as function of  $(\rho, \theta)$

Let us now return to the discussion of the general case and construct the extended Hamiltonian. Noting that for  $n \geq 2$ ,  $\chi_\rho^{(n)} = 0$ , so that

$$H_c^{(n+1)} = -\frac{1}{n+1} \int dx dy dz \varphi_\alpha(x) (\omega_{\alpha\beta})^{-1} (\sigma_{\beta\gamma})^{-1} G_\gamma^{(n)}, \quad (3.29)$$

where  $G_\gamma^{(n)}$  is given by

$$G_\gamma^{(n)} = \{\chi_\gamma, H_c^{(n)}\} \quad (3.30)$$

Since  $(\omega_{\alpha\beta})^{-1}$  and  $(\sigma_{\beta\gamma})^{-1}$  are proportional to Dirac delta functions, we have:

$$H_c^{(0)} = \int dx \left[ \frac{1}{2} \rho (\nabla \theta)^2 - V(\rho) \right] \quad (3.31)$$

so that

$$\begin{aligned} G_\rho^{(0)} &= \{\chi_\rho, H_c^{(0)}\} \\ &= -\frac{1}{2} (\nabla \theta)^2 + \partial_\rho V \end{aligned} \quad (3.32)$$

and also

$$\begin{aligned} G_\theta^{(0)} &= \{\chi_\theta, H_c^{(0)}\} \\ &= -\frac{1}{2} \rho \nabla^2 \theta \end{aligned} \quad (3.33)$$

so that the correction  $H_c^{(1)}$  is given by

$$\begin{aligned} H_c^{(1)} &= - \int dx \left\{ \left[ \frac{1}{2} (\nabla \theta)^2 - \partial_\rho V \right] \varphi_\rho \right. \\ &\quad \left. - \frac{1}{2} (\rho \nabla^2 \theta) \varphi_\theta \right\} \end{aligned} \quad (3.34)$$

Continuing the iteration process and summing up all the contributions we find that the canonical Hamiltonian is then given by (see the Appendix A)

$$\begin{aligned} H_c &= H_c^{(0)} + \int dx \left[ -\lambda_\theta \varphi_\rho + \lambda_\rho \varphi_\theta - \frac{1}{2} (\partial_\rho^2 V) \varphi_\rho^2 \right. \\ &\quad \left. + \dots + \frac{(-1)^n}{n!} (\partial_\rho^n V) \varphi_\rho^n + \dots \right] \\ &= H_c^{(0)} + \int dx \left[ -\lambda_\theta \varphi_\rho + \lambda_\rho \varphi_\theta \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{(-1)^{n+2}}{(n+2)!} (\partial_\rho^{n+2} V) \varphi_\rho^{(n+2)} \right], \end{aligned} \quad (3.35)$$

where the term corresponding to  $H_c^{(1)}$  is contained in  $\lambda_\theta = \frac{1}{2} (\nabla \theta)^2 - \partial_\rho V$ . Note that from the definition (3.6) the first-class constraints are given by

$$\Omega_\rho = \chi_\rho + \sigma_{\rho\theta} \varphi^\theta \quad (3.36)$$

$$\Omega_\theta = \chi_\theta + \sigma_{\theta\rho} \varphi^\rho \quad (3.37)$$

where  $\sigma_{\rho\theta} = -\sigma_{\theta\rho} = 1$ . Let us now go back to the canonical Hamiltonian Eq. (3.35) and analyze its last term. We note that

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(-1)^{n+2}}{(n+2)!} (\partial_{\rho}^{(n+2)} V) \varphi_{\rho}^{(n+2)} \\ &= \sum_{n=0}^{\infty} \Theta(n-1) \frac{(-1)^n}{n!} (\partial_{\rho}^n V) \varphi_{\rho}^n \\ &= -V(\rho) + \varphi_{\rho} \partial_{\rho} V(\rho) - e^{-\varphi_{\rho} \partial_{\rho}} V(\rho), \end{aligned} \quad (3.38)$$

where  $\Theta(x)$  is the Heavside function. This way we have

$$\begin{aligned} H_c &= \int dx [-\lambda_{\theta} \varphi_{\rho} + \lambda_{\rho} \varphi_{\theta} - e^{-\varphi_{\rho} \partial_{\rho}} V(\rho)] \\ &= \int dx \left[ \frac{1}{2} \rho (\nabla \theta)^2 - \frac{1}{2} (\nabla \theta)^2 \varphi_{\rho} \right. \\ &\quad \left. - \frac{1}{2} \rho \nabla^2 \theta \varphi_{\theta} - e^{-\varphi_{\rho} \partial_{\rho}} V(\rho) \right]. \end{aligned} \quad (3.39)$$

In order to find the corresponding Lagrangian, we identify  $\varphi \equiv \varphi_{\rho}$  and  $\Pi_{\varphi} \equiv \varphi_{\theta}$  as a pair of canonical conjugate coordinates and write the generating functional

$$\begin{aligned} \mathcal{Z} &= \mathcal{N} \int [d\rho][d\Pi_{\rho}][d\theta][d\Pi_{\theta}][d\varphi][d\Pi_{\varphi}] \\ &\quad \times \delta(\Pi_{\rho} - \theta + \Pi_{\varphi}) \delta(\Pi_{\theta} - \varphi) \\ &\quad \times \exp \left\{ i \int dx \left[ \Pi_{\rho} \dot{\rho} + \Pi_{\theta} \dot{\theta} + \Pi_{\varphi} \dot{\varphi} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \rho (\nabla \theta)^2 + \frac{1}{2} (\nabla \theta)^2 \varphi \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \rho \nabla^2 \theta \Pi_{\varphi} + e^{-\varphi \partial_{\rho}} V(\rho) \right] \right\} \end{aligned} \quad (3.40)$$

where the delta functions represent the first-class constraints. Noting that

$$\begin{aligned} \text{(i)} \quad & \int [d\Pi_{\theta}] \delta(\Pi_{\theta} - \varphi) \exp \left\{ i \int dx \Pi_{\theta} \dot{\theta} \right\} \\ &= \exp \left\{ i \int dx \varphi \dot{\theta} \right\} \\ \text{(ii)} \quad & \int [d\Pi_{\rho}] \delta(\Pi_{\rho} - \theta - \Pi_{\varphi}) \exp \left\{ i \int dx \Pi_{\rho} \dot{\rho} \right\} \\ &= \exp \left\{ i \int dx (\Pi_{\varphi} + \theta) \dot{\rho} \right\} \end{aligned}$$

and substituting these results into the generating functional we find

$$\begin{aligned} \mathcal{Z} &= \mathcal{N} \int [d\rho][d\theta][d\varphi][d\Pi_{\varphi}] \\ &\quad \exp \left\{ i \int dx \left[ \theta \dot{\rho} + \Pi_{\theta} \dot{\theta} + \Pi_{\varphi} \dot{\varphi} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \rho (\nabla \theta)^2 + \frac{1}{2} (\nabla \theta)^2 \varphi \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \rho \nabla^2 \theta \Pi_{\varphi} + e^{-\varphi \partial_{\rho}} V(\rho) \right] \right\}. \end{aligned} \quad (3.41)$$

The functional integral over  $\Pi_{\varphi}$  gives

$$\begin{aligned} & \int [d\Pi_{\varphi}] \exp \left\{ i \int dx \Pi_{\varphi} (\dot{\rho} + \dot{\varphi} + \frac{1}{2} \rho \nabla^2 \theta) \right\} \\ &= \delta(\dot{\rho} + \dot{\varphi} + \lambda_{\rho}) \end{aligned}$$

so that the generating functional reads

$$\begin{aligned} \mathcal{Z} &= \mathcal{N} \int [d\rho][d\theta][d\varphi] \delta(\dot{\rho} + \dot{\varphi} + \lambda_{\rho}) \\ &\quad \times \exp \left\{ i \int dx \left[ \theta \dot{\rho} + \varphi \dot{\theta} - \frac{1}{2} (\rho - \varphi) (\nabla \theta)^2 \right. \right. \\ &\quad \left. \left. + e^{-\varphi \partial_{\rho}} V(\rho) \right] \right\} \end{aligned} \quad (3.42)$$

and then the extended Lagrangian density is given by

$$\tilde{\mathcal{L}} = \theta \dot{\rho} + \varphi \dot{\theta} - \frac{1}{2} (\rho - \varphi) (\nabla \theta)^2 + e^{-\varphi \partial_{\rho}} V(\rho). \quad (3.43)$$

In the limit in which the auxiliary field vanishes,  $\varphi \rightarrow 0$ , we get back the original Lagrangian (2.1) as it should. Let us now look at the last term of this Lagrangian which involves the potential  $V(\rho)$ . Assuming that the potential function can be expanded in a power series we have

$$e^{-\varphi \partial_{\rho}} V(\rho) = V(\rho - \varphi) \quad (3.44)$$

So, we can apply the above equation for a wide range of functions  $V(\rho)$ , as for example, the potentials discussed in section II.

Then, the extended Lagrangian density after a partial integration becomes

$$\tilde{\mathcal{L}} = -(\rho - \varphi) \dot{\theta} - \frac{1}{2} (\rho - \varphi) (\nabla \theta)^2 + V(\rho - \varphi) \quad (3.45)$$

so that the Lagrangian is invariant under the exchange  $\rho \rightarrow \rho - \varphi$  which is the gauge symmetry of the model. Since the first-class  $\Omega$  is strongly involutive with canonical Hamiltonian (see the Appendix B) it is easy to check the invariance of  $\tilde{\mathcal{L}}(\rho, \theta, \varphi)$ .

Now, we can look at the consequences of this gauge symmetry on the previous known symmetries for the fluid dynamical model. The Galileo and Poincaré groups in the gauged model can be obtained from the the density generators of the non-gauge model, Eqs. (2.4)-(2.8), (2.11) and (2.12) simply through the shift  $\tilde{\mathcal{O}} = e^{-\varphi \partial_{\rho}} \mathcal{O}$ , so that the original Galileo and Poincaré invariances of the fluid field model are preserved by the introduction of the auxiliary field which bring to it a gauge symmetry.

#### IV. CONCLUSIONS

In this article we have studied the fluid field theory [9] for which we found a class of classical solutions which recover previous known particular cases [10].

Then, by means of the BFFT formalism [4] we extended the original phase space by including new fields

which permitted the transformation of the set of second-class constraints into a first-class one.

We have analyzed a situation where we found an extended gauge symmetry for an arbitrary potential with linear constraints and a kind of a Wess-Zumino Lagrangian was built. As a result we have obtained a new gauge invariant system. This new system may be of interest to the membrane problem related to Lagrangian (2.1) since that formulation corresponds to a gauge fixed version in the light-cone gauge.

As a final remark it is important to mention that the procedure discussed here could also be applied successfully to a situation where nonlinear constraints were involved, as is well known in general for the BFFT method.

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## APPENDIX A

In this Appendix we give some details of the iteration process necessary to construct the canonical Hamiltonian in the case of the linear constraint discussed in section III. Using Eqs. (3.29) and (3.30) we found the first correction,  $H_c^{(1)}$ . For the next correction we have:

$$\begin{aligned} G_\rho^{(1)} &= \{\chi_\rho, H_c^{(1)}\} \\ &= -\frac{\delta}{\delta\rho} H_c^{(1)} \\ &= -(\partial_\rho^2 V)\varphi_\rho - \frac{1}{2}(\nabla^2\theta)\varphi_\theta \end{aligned} \quad (\text{A.1})$$

while

$$\begin{aligned} G_\theta^{(1)} &= \{\chi_\theta, H_c^{(1)}\} \\ &= -\frac{\delta}{\delta\theta} H_c^{(1)} \\ &= +\frac{1}{2}(\nabla^2\theta)\varphi_\rho \end{aligned} \quad (\text{A.2})$$

so that

$$H_c^{(2)} = -\frac{1}{2} \int dx (\partial_\rho^2 V)\varphi_\rho^2. \quad (\text{A.3})$$

For third correction to the canonical Hamiltonian, we find

$$\begin{aligned} G_\rho^{(2)} &= \{\chi_\rho, H_c^{(2)}\} \\ &= -\frac{\delta}{\delta\rho} H_c^{(2)} \\ &= \frac{1}{2}(\partial_\rho^3 V)\varphi_\rho^2 \end{aligned} \quad (\text{A.4})$$

and

$$G_\theta^{(2)} = -\frac{\delta}{\delta\theta} H_c^{(1)} = 0 \quad (\text{A.5})$$

so that

$$H_c^{(3)} = \frac{1}{2.3} \int dx (\partial_\rho^3 V)\varphi_\rho^3. \quad (\text{A.6})$$

For the next term we have

$$\begin{aligned} G_\rho^{(3)} &= \{\chi_\rho, H_c^{(3)}\} \\ &= -\frac{\delta}{\delta\rho} H_c^{(3)} \\ &= \frac{1}{2.3}(\partial_\rho^4 V)\varphi_\rho^3 \end{aligned} \quad (\text{A.7})$$

and  $G_\theta^{(n)} = 0$  for  $n \geq 2$  so that

$$H_c^{(4)} = -\frac{1}{2.3.4} \int dx (\partial_\rho^4 V)\varphi_\rho^4. \quad (\text{A.8})$$

## APPENDIX B

Let us show here that the first-class constraints  $\Omega_\alpha$  are strongly involutive in respect to the canonical Hamiltonian,  $H_c(\rho, \theta, \varphi_\rho, \varphi_\theta)$ , i. e.,  $\{\Omega_\alpha, H_c\} = 0$ .

First note that from definition of  $\Omega_\rho$ ,  $\Omega_\theta$ , Eqs. (3.36), (3.37), and the canonical Hamiltonian Eq. (3.39), we have

$$\begin{aligned} \{\Omega_\rho, H_c\} &= \{\Omega_\rho, H_c^{(0)}\} + \{\lambda_\rho, \Omega_\rho\}\varphi_\theta - \lambda_\theta\{\Omega_\rho, \varphi_\rho\} \\ &\quad - \{\lambda_\theta, \Omega_\rho\}\varphi_\rho - \{\Omega_\rho, e^{-\varphi_\rho\partial_\rho} V\}, \end{aligned} \quad (\text{B.1})$$

where we have used the fact that  $\{\Omega_\rho, \varphi_\theta\} = 0$ . Then,

$$\{\Omega_\rho, H_c\} = \{\lambda_\rho, \Omega_\rho\}\varphi_\theta - \{\lambda_\theta, \Omega_\rho\}\varphi_\rho. \quad (\text{B.2})$$

Now, using  $\varphi_\theta = \Omega_\rho - \chi_\rho$  and  $\varphi_\rho = \Omega_\theta - \chi_\theta$  we have

$$\begin{aligned} \{\Omega_\rho, H_c\} &= \{\lambda_\rho, \Omega_\rho\}(\Omega_\rho - \chi_\rho) - \{\lambda_\theta, \Omega_\rho\}(\Omega_\theta - \chi_\theta) \\ &= (\{\lambda_\rho, \Omega_\rho\} - \{\lambda_\rho, \chi_\rho\})\Omega_\rho \\ &\quad + (\{\lambda_\theta, \Omega_\theta\} - \{\lambda_\theta, \chi_\theta\})\Omega_\rho \\ &= \{\lambda_\rho, \Omega_\rho\}\Omega_\rho + \{\lambda_\theta, \Omega_\theta\}\Omega_\rho \end{aligned} \quad (\text{B.3})$$

and then we find

$$\{\Omega_\rho, H_c\} = 0. \quad (\text{B.4})$$

An analogous result can be found for  $\Omega_\theta$ , proving our original statement.

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